Sobolev Integral Probability Metric

$$d_{\mathcal{F}}(\mu,\nu) = \sup_{f\in\mathcal{F}} \mathbb{E}_{x\sim\mu}(f(x)) - \mathbb{E}_{x\sim\nu}(f(x))$$

$\mathcal{F} = \mathrm{Lip}_1$ $\quad W_1(\mu,\nu) = \sup_{f\in \mathrm{Lip}_1} \int f \, d(\mu-\nu)$

$$L^2 = \{ f : \left(\int |f|^2\right)^{1/2} < \infty \}$$

$$\|f\|_2 = \left(\int |f|^2\right)^{1/2}$$

$$B_2 = \{ f : \|f\| \leq 1 \}$$

Fischer IPM

$\mathcal{F} = B_2$ $\quad\quad$ μ, ν on $\Omega \subset \mathbb{R}^d$

$$F_2(\mu, \nu) = \sup_{f \in B_2} \int f \, d(\mu - \nu)$$

$\mu, \nu << \lambda^d$

$$\frac{d\mu}{d\lambda} = f(x)$$

$$\frac{d\nu}{d\lambda} = g(x)$$

$\varphi \in B_2$

$$\int_\Omega \varphi \, d(\mu - \nu)$$

$$= \int_\Omega \varphi \, (f - g) \, dx$$

$$= \langle \varphi, f - g \rangle$$

$$\sup_{\|f\| \leq 1} \langle g, f \rangle = \|g\|$$

$$\sup_{\varphi \in B_2} \int \varphi \, d(\mu - \nu)$$

$$= \sup_{\varphi \in B_2} \langle \varphi, f - g \rangle$$

$$= \|f - g\|_2 = F_2(\mu, \nu)$$

Sobolev space $W_0^{1,2}(\Omega)$

$f \in W_0^{1,2}$

(1) $\varphi \in C^1$ $\displaystyle\int_\Omega f \frac{\partial \varphi}{\partial x_i} = - \int_\Omega \mu \varphi \, dx$

$u = Df$

(2) $\|f\|_{W_0^{1,2}}$

$$= \left(\int_\Omega |f|^2 + |Df|^2 \right)^{1/2}$$

$$= C \left(\int |Df|^2 \right)^{1/2} = C \|Df\|_{L^2}$$

(3) $u \in W_0^{1,2} \Rightarrow u = 0$ on $\partial\Omega$

$$\mathcal{F} = \{ f \in W_0^{1,2}(\Omega) : \|f\|_{W_0^{1,2}} \le 1 \}$$

$\mu, \nu \xrightarrow{\text{CDF}}$

$$F_{\mu}(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(x)\, dx$$

$F_\mu(x) \in C^d$

$$D^{-i} = \frac{\partial^{d-1}}{\partial x_1 \cdots \partial_{i-1} \partial_{i+1} \cdots \partial_d}$$

$$D^- = (D^{-1}, D^{-2}, \ldots, D^{-d})$$

$$\varphi \in B_2$$

$$\int_\Omega \varphi \, d(\mu - \nu) = \int_\Omega \varphi \, D(F_\mu(x) - F_\nu(x)) \, dx$$

$$= \int \varphi \frac{\partial}{\partial x_i} D^{-i}(F_\mu - F_\nu) \, dx$$

$$= -\int \frac{\partial \varphi}{\partial x_i} \cdot D^{-i}(F_\mu - F_\nu)$$

$$= \int \frac{\partial \varphi}{\partial x_i} D^{-i}(F_\nu - F_\mu)$$

$$\int \varphi \cdot d(\mu - \nu) = \frac{1}{d} \sum_{i=1}^{d} \int_\Omega \frac{\partial \varphi}{\partial x_i} \cdot D^{-i}(F_\mu - F_\nu)$$

$$= \frac{1}{d} \int_\Omega \langle \nabla \varphi, D^-(F_\mu - F_\nu) \rangle$$

$$L^2(\Omega; \mathbb{R}^d)$$

$$\langle f, g \rangle_{L^2(\Omega;\mathbb{R}^d)} = \int_\Omega \langle f, g \rangle \, dx$$

$$\langle f, f \rangle = \|f\|^2$$

$$\varphi \in B_2$$

$$\int \varphi \, d(\mu - \nu) = \frac{1}{d} \langle \nabla \cdot \varphi, D^-(F_\nu - F_\mu) \rangle$$

$$\sup_{\varphi \in B_2} \int \varphi \, d(\mu - \nu) = \underline{\sup_{\varphi \in B_2} \frac{1}{d}} \langle \nabla \varphi, D^-(F_\nu - F_\mu) \rangle$$

$\tilde{B}_2$ = 1 ball in $L^2(\Omega; \mathbb{R}^d)$

$$= \sup_{\psi \in \tilde{B}_2} \frac{1}{d} \langle \psi, D^-(F_\nu - F_\mu) \rangle$$

$$= \frac{1}{d} \| D^-(F_\nu - F_\mu) \|_{L^2(\Omega;\mathbb{R}^d)}$$

How does Sobolev IPM relate to W_1?

$$W_1 = \sup_{\varphi \in Lip_1} \int f \, d(\mu - \nu)$$

$$S_2(\mu, \nu) = \sup_{\varphi \in B_2} \int \varphi \, d(\mu - \nu)$$

$\nabla \varphi$ exists a.e

$|\nabla \varphi| \leq 1$ a.e

$\Omega \subset \mathbb{R}^d$ compact

$diam(\Omega) < \infty$

$\varphi \in Lip_1(\Omega)$ $\quad \left\| \frac{\nabla \varphi}{diam(\Omega)} \right\|_{L^2} \leq 1$

$$diam(\Omega)^{-1} W_1(\mu, \nu) \leq S_2(\mu, \nu)$$

If μ, ν are abs cont,
with $|f|, |g| < C \in \mathbb{R}$

$\varphi \in W_0^{1,2}(\Omega)$

$$\int \varphi \, d(\mu - \nu) \leq \sqrt{C} \, \underbrace{\|\nabla \varphi\|_{L^2}}_{} W_2(\mu, \nu)$$

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$S_2(\mu, \nu)$

$$= \sup_{\varphi \in B_2} \int \varphi \, d(\mu - \nu) \leq \sqrt{C} \cdot W_2(\mu, \nu)$$

$\forall \varphi \in B_2 : \|\nabla \varphi\|_{L^2} \leq 1$

$$\operatorname{diam}(\Omega)^{-1} W_1(\mu, \nu) \leq S_2(\mu, \nu)$$
$$\leq \sqrt{C} \, W_2(\mu, \nu)$$